

SANDPILES, SPANNING TREES, AND PLANE DUALITY

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ABSTRACT. Let G be a connected, loopless multigraph. The sandpile group of G is a finite abelian group associated to G whose order is equal to the number of spanning trees in G . Holroyd et al. used a dynamical process on graphs called rotor-routing to define a simply transitive action of the sandpile group of G on its set of spanning trees. Their definition depends on two pieces of auxiliary data: a choice of a ribbon graph structure on G , and a choice of a root vertex. Chan, Church, and Grochow showed that if G is a planar ribbon graph, it has a canonical rotor-routing action associated to it, i.e., the rotor-routing action is actually independent of the choice of root vertex.

It is well-known that the spanning trees of a planar graph G are in canonical bijection with those of its planar dual G^* , and furthermore that the sandpile groups of G and G^* are isomorphic. Thus, one can ask: are the two rotor-routing actions, of the sandpile group of G on its spanning trees, and of the sandpile group of G^* on its spanning trees, compatible under plane duality? In this paper, we give an affirmative answer to this question, which had been conjectured by Baker.

1. INTRODUCTION

Let G be a connected multigraph with no loop edges. The *sandpile group* of G is a finite abelian group whose order is equal to the number of spanning trees in G ; it is the group of *degree zero divisors* of G modulo the equivalence relation generated by *lending moves*. (We will recall all relevant definitions in Section 2.)

In [7], Holroyd, Levine, Meszarós, Peres, Propp, and Wilson use a dynamical process on graphs called *rotor-routing* to define a simply transitive action of the sandpile group of G on its set of spanning trees. Rotor-routing itself was introduced in [9] under the name “Eulerian walkers” and has been rediscovered several times in different fields: see [7] for a concise history of the topic.

The definition of the rotor-routing action on G given in [7] involves two pieces of auxiliary data. First, the action is defined with respect to a choice of a root vertex $v \in V(G)$, or *basepoint*. Second, it depends on a *ribbon graph* structure on G : a choice of a cyclic ordering of the set of edges incident to each vertex v . Note that such a choice of cyclic orders defines an embedding of G on some closed, oriented surface S , in which all cyclic orders correspond to a positive orientation, say, with respect to S . We say that G is a *planar*

ribbon graph if S is just a sphere, i.e., if the chosen ribbon structure equips G with an embedding into the plane.

A recent paper of Chan, Church, and Grochow [5] answers a question of J. Ellenberg [8] by proving that the rotor-routing action does not depend on the choice of basepoint if and only if G is a planar ribbon graph. This result is somewhat surprising, and as a nice consequence of it, we may henceforth refer to *the* rotor-routing action on a planar ribbon graph, without further reference to a choice of basepoint.

Any graph G embedded in the plane has a planar dual graph G^* whose spanning trees are in canonical bijection with those of G . Moreover, the sandpile groups of G and G^* are, up to sign, canonically isomorphic [1] (see also [6]). Thus, one would hope that the two rotor-routing actions, of the sandpile group of G on the set $\mathcal{T}(G)$ of its spanning trees, and of the sandpile group of G^* on its spanning trees, are compatible.

This was, in fact, exactly the conjecture suggested to us by M. Baker. In this paper, we provide a proof of Baker’s conjecture on the compatibility of the rotor-routing action of the sandpile group with plane duality. See Theorem 3.1 for the precise statement, and see Figure 1 for an example illustrating the result.

We begin with preliminary definitions on the sandpile group and rotor-routing in Section 2. The proof of our main result occupies Section 3. The key idea of our proof is the *angle* between two spanning trees T and T' of G : see Definition 3.3. The angle from T to T' remembers the element of the sandpile group that takes T to T' under rotor-routing. On the other hand, we are able to show using a direct geometric argument that the angle is compatible with plane duality, so the main theorem follows.

We would also like to refer the reader to the recent preprint [3], which arrives at another proof of Theorem 3.1 via a completely different route. In that paper, Baker and Wang prove that the bijections obtained by Bernardi in [4, Theorem 45] give rise to another simply transitive action of the sandpile group on the spanning trees of a ribbon graph G with a fixed root vertex. They show that this action is compatible with plane duality and that it coincides with the rotor-routing action when G is planar. It would be interesting to study the relationship between these two approaches further.

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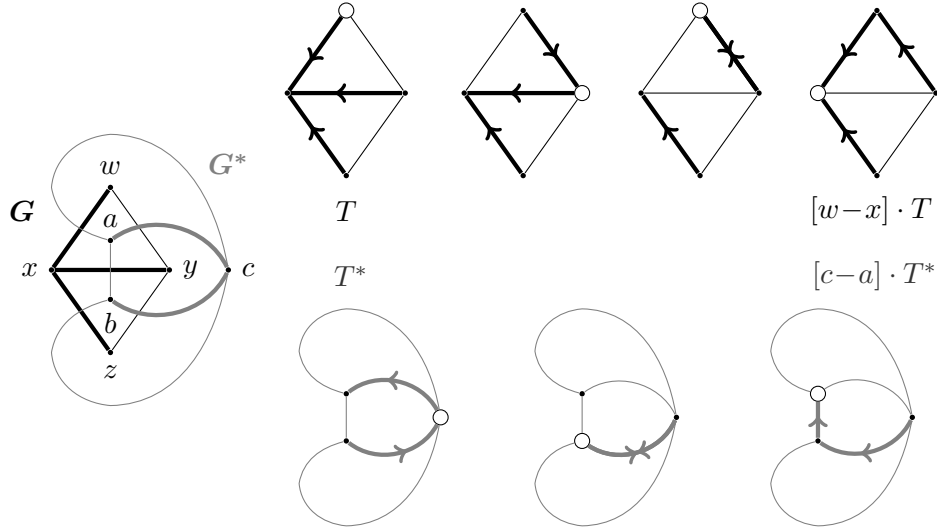


FIGURE 1. This figure shows the result of applying the element $[w-x]$ of $\mathcal{S}(G)$ to the spanning tree T and, in the bottom row, the result of applying the element $[c-a]$ of $\mathcal{S}(G^*)$ to T^* . The graph G has all rotors oriented clockwise relative to the page, and its planar dual G^* has all rotors oriented counterclockwise. We chose x and a as our basepoints of G and G^* for the respective computations. The isomorphism $\mathcal{S}(G) \cong \mathcal{S}(G^*)$ identifies $[w-x]$ and $[c-a]$, so the trees $[w-x] \cdot T$ and $[c-a] \cdot T^*$ must be dual trees, as shown on the right.

2. PRELIMINARIES

2.1. The sandpile group. Let $G = (V, E)$ be a finite connected loopless multigraph with vertex set V and edge multiset E . The set of *divisors* on G is the free abelian group on the vertices: $\text{Div}(G) = \mathbb{Z}V$. We imagine a divisor $D = \sum_{v \in V} a_v v$ to be an assignment of $D(v) := a_v$ chips to each vertex v , keeping in mind that this number may be negative. We write $\text{Div}^0(G)$ for the subgroup of divisors whose net number of chips $\sum D(v)$ is zero.

A *lending move* by a vertex v consists of removing $\deg(v)$ chips from v and distributing them along incident edges to the vertices neighboring v . In other words, letting $n(v, w)$ denote the number of edges between v and w , a lending move by v performed on a divisor D produces a divisor D' given by

$$D'(w) = \begin{cases} D(w) + n(v, w) & \text{if } w \neq v \\ D(v) - \deg(v) & \text{if } w = v. \end{cases}$$

Notice that lending moves do not change the total number of chips in a divisor. Divisors D and D' are *linearly equivalent*, denoted $D \sim D'$, if one can be obtained from the other by a sequence of lending moves at various

vertices. The *sandpile group* of G is

$$\mathcal{S}(G) = \text{Div}^0(G)/\sim.$$

The sandpile group of a graph is also variously known as the *Jacobian* of G , the *Picard group* $\text{Pic}^0(G)$, or the *critical group* of G .

2.2. Integral cuts and cycles. Fix an arbitrary orientation on the edges E , and let $\mathbb{Z}E$ be the free abelian group on these oriented edges. If $e = \{u, v\} \in E$ is given the orientation (u, v) , we write $e^+ = \text{head}(e) = v$ and $e^- = \text{tail}(e) = u$. We identify $-e$ with the oppositely oriented edge (v, u) . Each directed cycle on the underlying undirected graph G may be thought of as an element of $\mathbb{Z}E$, and the \mathbb{Z} -linear span of these cycles in $\mathbb{Z}E$ is the *integral cycle space* for G , which we denote by \mathcal{C} .

Next, for any subset $U \subset V$, the collection of all edges joining a vertex of U to a vertex of $V \setminus U$ is called a *cut*. By directing all of these edges from vertices in U to vertices in $V \setminus U$, we can identify this cut with an element of $\mathbb{Z}E$. If U consists of single vertex v , this cut is called a *vertex cut* at v . The integer span of all cuts is the *integral cut space* for G and is denoted by \mathcal{C}^* . Note that the vertex cuts generate the cut space.

Define

$$\mathcal{E}(G) = \mathbb{Z}E/(\mathcal{C} + \mathcal{C}^*).$$

We now identify $\mathcal{E}(G)$ with the sandpile group $\mathcal{S}(G)$, as follows. Define the *boundary map* $\mathbb{Z}E \rightarrow \text{Div}^0(G)$ by sending each edge e to $e^+ - e^-$. The boundary map is surjective since G is connected, and its kernel is exactly the cycle space of G , so it identifies $\text{Div}^0(G)$ with $\mathbb{Z}E/\mathcal{C}$. Now given $D \in \text{Div}^0(G)$, let D_v be the boundary of a vertex cut at the vertex v . Then $D + D_v$ is the divisor obtained from D by performing a lending move at v . Therefore the boundary map induces an isomorphism

$$\begin{aligned} \partial_G: \mathcal{E}(G) &\xrightarrow{\cong} \mathcal{S}(G) \\ e &\mapsto [e^+ - e^-], \end{aligned}$$

as was proved in [1, Proposition 8]. We will sometimes write ∂ instead of ∂_G for short.

2.3. Rotor-routing action on spanning trees. Fix a *ribbon graph* structure on G , i.e., for each vertex v , fix a cyclic ordering of the edges incident to v . Fix a vertex q . A *rotor configuration with basepoint q* is the choice for each vertex $v \neq q$ of an edge, $\rho(v)$, incident to v . We orient each edge $\rho(v)$ so that its tail is v .

Let D be a divisor on G , thought of as a chip configuration on G , and let ρ be a rotor configuration with basepoint q . We now recall the *rotor-routing* process, by which a divisor D transforms ρ into a new rotor configuration ρ' . *Firing* a vertex v consists of updating ρ by replacing $\rho(v)$ with the next edge in the cyclic ordering of edges at v , then removing a chip from v and placing it at the other end of the new edge $\rho(v)$. Note that firing v a total of $\deg(v)$

times does not change the original rotor configuration, but transforms D by a lending move at v . Now, every divisor D on G is linearly equivalent to a divisor D' with $D'(v) \geq 0$ for all $v \neq q$, see e.g. [2, Proposition 3.1]. From that point, [7] shows that solely through vertex firings, all chips may be routed into q , and the rotor configuration at the end of this process depends solely on the divisor class of D .

Let $\mathcal{T}(G)$ denote the set of spanning trees of G . Rooting $T \in \mathcal{T}(G)$ at q uniquely determines a rotor configuration ρ_T : for each vertex $v \neq q$, set $\rho_T(v)$ to be the edge incident to v on the path in T from v to q . Given a divisor class $[D] \in \mathcal{S}(G)$, use the rotor routing process to route all chips into q (at which point, all chips will be gone since $\deg(D) = 0$). It is shown in [7] that the resulting rotor configuration is a spanning tree, directed into q . Call the underlying undirected spanning tree $[D] \cdot T$. Then according to [7], the resulting map

$$\begin{aligned} \mu_G: \mathcal{S}(G) \times \mathcal{T}(G) &\rightarrow \mathcal{T}(G) \\ ([D], T) &\mapsto [D] \cdot T \end{aligned}$$

is a simply transitive action of $\mathcal{S}(G)$ on $\mathcal{T}(G)$.

2.4. Planar duality. Now suppose that $G = (V, E)$ is a planar ribbon graph, and let $G^* = (V^*, E^*)$ be its *planar dual graph*, whose vertices are the faces of G and whose edges cross the edges of G . We shall assume throughout that both G and G^* are loopless, i.e., G has neither bridges nor loops. We write e^* for the edge of G^* crossing the edge e of G . Each spanning tree of G determines a spanning tree of G^* : namely, there is a natural bijection

$$\delta: \mathcal{T}(G) \xrightarrow{\cong} \mathcal{T}(G^*)$$

sending T to the tree $T^* = \{e^* \in E^* : e \in E \setminus T\}$.

Let us call the orientation of the plane that agrees with the cyclic orderings of G *clockwise*. Then we fix once and for all the following *planar dual ribbon graph structure* on G^* : take the cyclic orderings of the edges at the vertices of G^* to be *counter-clockwise* with respect to the plane.

In order to define $\mathbb{Z}E$, we fixed an arbitrary orientation of the edges of G . To define $\mathbb{Z}E^*$, we will now choose a compatible orientation on the edges of G^* . For an oriented edge e of G , let e' (respectively e'') denote the edge at $v = e^-$ before (respectively after) e in the cyclic order at v . Now, call the face between e' and e at v the face *before* e , and call the face between e and e'' at v the face *after* e . Then we orient e^* so that its head is the face of G before e , and its tail is the face of G after e . For example, in Figure 1, with the rotors of G oriented clockwise relative to the page, suppose e is the directed edge from x to y . Then e^* is the directed edge in G^* from b to a .

Since directed cycles of G are directed cuts of G^* and vice versa, mapping each edge to its dual produces an isomorphism $\mathcal{E}(G) \cong \mathcal{E}(G^*)$, and hence we get an isomorphism ϕ of sandpile groups labeled as in the following

commutative diagram:

$$\begin{array}{ccc} \mathcal{E}(G) & \xrightarrow{\cong} & \mathcal{E}(G^*) \\ \partial_G \downarrow & & \downarrow \partial_{G^*} \\ \mathcal{S}(G) & \xrightarrow{\phi} & \mathcal{S}(G^*). \end{array}$$

3. COMPATIBILITY OF ROTOR-ROUTING WITH DUALITY

Let G be any planar ribbon graph such that both G and its dual G^* are loopless. In the previous section, we established an isomorphism $\phi: \mathcal{S}(G) \rightarrow \mathcal{S}(G^*)$ that depended on a single global choice of orientation of the E^* derived from the orientation E . With respect to this choice, we may now state the main theorem of the paper:

Theorem 3.1. *The diagram*

$$\begin{array}{ccc} \mathcal{S}(G) \times \mathcal{T}(G) & \xrightarrow{\mu_G} & \mathcal{T}(G) \\ \phi \times \delta \downarrow & & \downarrow \delta \\ \mathcal{S}(G^*) \times \mathcal{T}(G^*) & \xrightarrow{\mu_{G^*}} & \mathcal{T}(G^*) \end{array}$$

commutes. In other words, the rotor-routing action is compatible with plane duality.

In the rest of this section, we prove Theorem 3.1. We begin with a topological definition of the angle between two spanning trees; this definition applies to all ribbon graphs, not just planar ones, and is the key idea in our proof of Theorem 3.1.

Suppose G is any ribbon graph, and let e and e' be directed edges emanating from a vertex u . Suppose that in the cyclic order starting from $e = e_0$, the edges between e and e' are e_0, e_1, \dots, e_k where $e_k = e'$, all directed outward from u . Define the *angle* between e and e' at u by

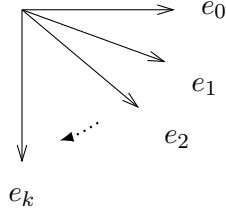
$$\angle^u(e, e') = \sum_{i=1}^k \partial e_i \in \mathcal{S}(G).$$

Recall that ∂ denotes the boundary map sending a directed edge e to the element $[e^+ - e^-] \in \mathcal{S}(G)$. Note that the sum includes e' but not e .

Lemma 3.2. *Suppose G is a planar ribbon graph, and let e_0, \dots, e_k be consecutive outgoing edges from some vertex u in the cyclic order at u . For $i = 0, \dots, k$, let r_i be the face of G , equivalently the vertex of G^* , lying to the right of e_i (with respect to the cyclic order at u). Then*

$$\phi(\angle^u(e_0, e_k)) = [r_0 - r_k] \in \mathcal{S}(G^*).$$

Proof. By definition, $\phi(\partial e_i) = [r_{i-1} - r_i]$. By linearity, it follows that $\phi(\angle^u(e_0, e_k))$ is the telescoping sum $[(r_0 - r_1) + (r_1 - r_2) + \dots + (r_{k-1} - r_k)]$, proving the claim. \square

FIGURE 2. $\angle^u(e_0, e_k) = \partial e_1 + \cdots + \partial e_k$.

Definition 3.3. Let G be an arbitrary ribbon graph, and let T and T' be two spanning trees of G . Let $v \in V$ be any vertex. As in §2.3, let ρ_T and $\rho_{T'}$ be the rotor configurations based at v arising from orienting T and T' towards v .

The *angle* between T and T' based at v , denoted $\angle_v(T, T')$, is the sum of the angles between their edges at each non-root vertex. That is,

$$\angle_v(T, T') := \sum_{u \in V \setminus \{v\}} \angle^u(\rho_T(u), \rho_{T'}(u)) \in \mathcal{S}(G).$$

Lemma 3.4. Let G be any ribbon graph, and let T be a spanning tree of G . For any vertex v and any $[D] \in \mathcal{S}(G)$, we have

$$\angle_v(T, [D] \cdot T) = [-D].$$

Here, the rotor-routing action of $[D]$ on T is computed with respect to the basepoint v .

Proof. Without loss of generality, we may choose D to be a chip configuration that is nonnegative at vertices other than v . Consider the rotor-routing process that calculates $[D] \cdot T$. We will say that the directed edge (x, y) is *activated* if a chip is sent from vertex x to vertex y during this process. Note that when the chip is fired, the chip configuration on the graph changes by $\partial(x, y) = y - x$. Since at the end of the rotor-routing process there are no chips left on the graph, it follows that

$$[D] + \sum_e \partial e = 0,$$

where the sum is over the multiset of edges that have been activated during the process.

Next, we claim that the angle between T and $[D] \cdot T$ is in fact equal to $\sum_e \partial e$, where the sum is again over the multiset of activated edges. This is because at each vertex $u \neq v$, the sum of the boundaries of all outgoing edges e at u is $0 \in \mathcal{S}(G)$; after all, this sum corresponds to a lending move at u . So the sum over all activated edges leaving u is exactly the angle at u between the edge of T leaving u and that of T' , and the claim follows.

Summarizing, we have

$$\angle_v(T, [D] \cdot T) = \sum_e \partial e = [-D].$$

□

Corollary 3.5. *Let G be any planar ribbon graph, and let T and T' be spanning trees of G rooted at the same vertex v . Then $\angle_v(T, T') = 0$ if and only if $T = T'$.*

Proof. Assume that $\angle_v(T, T') = 0$, and let $[D] \in \mathcal{S}(G)$ take T to T' under the rotor-routing action with basepoint v . It follows from [7, Lemma 3.17] that the element $[D]$ exists and is unique. Then by Lemma 3.4, $[D] = 0$, so $T = T'$. The converse is clear. □

Remark 3.6. It follows from Lemma 3.4 and from [5, Theorem 2] that the notion of angle between trees for G is independent of the choice of root vertex for the trees if and only if G is a planar ribbon graph. Indeed, Lemma 3.4 shows that $\angle_v(T, T')$ is exactly the element of $\mathcal{S}(G)$ sending T' to T in the rotor-routing action based at v , and the rotor-routing action is basepoint-independent if and only if G is a planar ribbon graph by [5]. Thus, if G is planar, we will henceforth write $\angle(T, T')$ for the angle between T and T' , computed with respect to any vertex.

We can now prove our main lemma.

Lemma 3.7. *Let G be a planar ribbon graph, and let T and T' be spanning trees of G . Then*

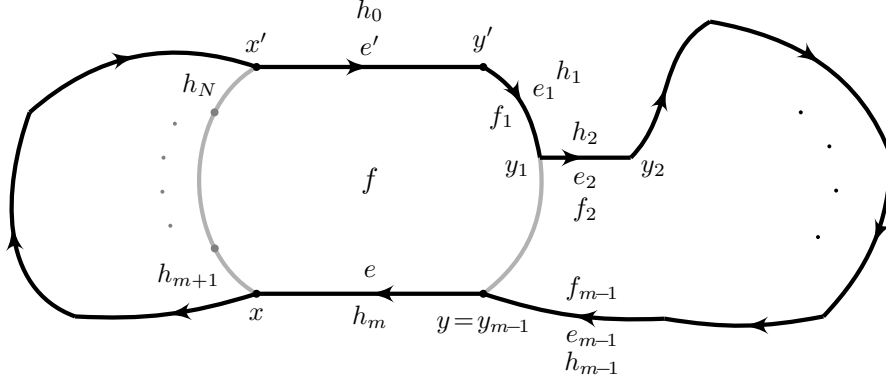
$$\phi(\angle(T, T')) = \angle(T^*, T'^*).$$

Proof. Given a spanning tree T and an edge e not in T , we call the unique cycle $C(e)$ in $T \cup \{e\}$ the *fundamental cycle* of e with respect to T . We first note that there is a sequence of trees $T = T_0, T_1, \dots, T_r = T'$ such that for each j , the trees T_{j+1} and T_j have exactly $n - 1$ edges in common. If $T = T'$ this statement is vacuously true. Otherwise, pick $e' \in T' \setminus T$; then the fundamental cycle of e' with respect to T must contain some edge $e \in T \setminus T'$. Set $T_1 = T \cup \{e'\} \setminus \{e\}$. Then T_1 and T' have smaller symmetric difference, so repeating, we produce a sequence of spanning trees as desired. It follows by induction that we may assume $T' = T \cup \{e'\} \setminus \{e\}$.

In fact, we may further assume, again by induction, that e and e' are edges incident to a common face of G . Indeed, since $T^* \cup \{e^*\} \setminus \{e'^*\} = T'^*$ is acyclic, the fundamental cycle $C(e^*)$ of e^* with respect to T^* contains e'^* . Now starting at e^* and proceeding along the cycle $C(e^*)$ in either direction, let $e^* = e_0^*, e_1^*, \dots, e_s^* = e'^*$ be the sequence of edges traversed. Then

$$T^*, (T^* \cup \{e^*\}) \setminus \{e_1^*\}, (T^* \cup \{e^*\}) \setminus \{e_2^*\}, \dots, (T^* \cup \{e^*\}) \setminus \{e_s^*\}$$

is a sequence of trees in G^* such that the symmetric difference of any consecutive pair of trees consists of two edges of G^* adjacent to the same vertex.

FIGURE 3. The fundamental cycle \mathcal{C} of $T \cup \{e'\}$, shaded in black.

Now passing to G , we conclude that

$$T, (T \cup \{e_1\}) \setminus \{e\}, (T \cup \{e_2\}) \setminus \{e\}, \dots, (T \cup \{e'\}) \setminus \{e\}$$

is a sequence of trees in G such that the symmetric difference of any consecutive pair of trees consists of two edges of G incident to the same face.

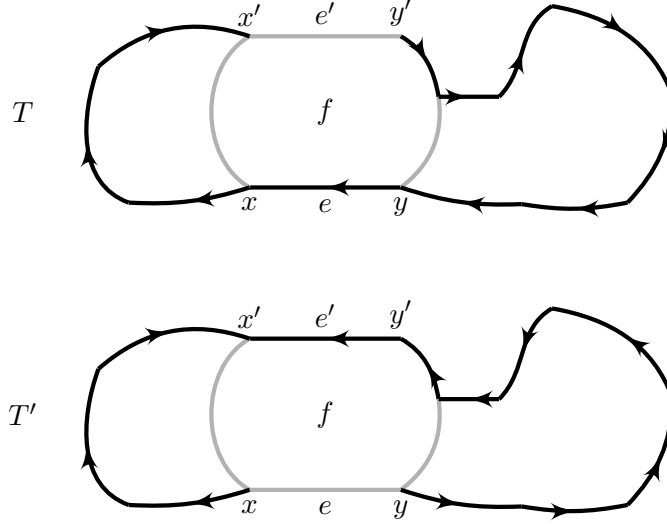
Thus from here on, we assume that $T' = (T \cup \{e'\}) \setminus \{e\}$, where $e, e' \in E(G)$ are incident to a common face, which we call f . Write $e = xy$ and $e' = x'y'$ for vertices x, y, x', y' of $V(G)$, such that f is to the left of the edge e when it is traversed in the direction $x \rightarrow y$, and f is to the right of the edge e' when it is traversed in the direction $x' \rightarrow y'$. Write \mathcal{C} for the fundamental cycle in $T \cup \{e'\}$; it is illustrated in Figure 3. (Here and throughout the rest of the proof, we assume a clockwise orientation on the rotors of G simply in order to talk about the left and right sides of an edge freely. For example, the face to the right of an oriented edge $e = (x, y)$ should be interpreted as the face coming in between e and the edge after e in the cyclic order at x .)

By Remark 3.6, the calculation of the angle $\angle(T, T') \in \mathcal{S}(G)$ is independent of the choice of root vertex. Choose x' as the root and orient T and T' towards x' . We wish to study the sum of the angles at each vertex $v \neq x'$ of G between the edges of T and T' that are outgoing from v .

Having rooted the trees at x' , we start by observing that the path between y and y' in T is directed from y' to y , whereas in T' it has the opposite orientation. This is illustrated in Figure 4. Furthermore, all other edges shared by T and T' have the same orientation. Indeed, consider a vertex v not on \mathcal{C} and say its unique path in T to x' first meets \mathcal{C} at v' ; then the same path $v-v'$ in T' must be an initial subpath of the unique path in T' from v to x' , so in particular the edge leaving v is unchanged.

Let us fix some notation before going further. Write

$$y' = y_0, e_1, y_1, e_2, \dots, y_{m-1} = y$$

FIGURE 4. Parts of the trees T and T' , rooted at the vertex x' .

for the sequence of vertices and directed edges in the $y'-y$ path in T . For each directed edge e_i , we write f_i (respectively h_i) for the face of G to the right (respectively left) of e_i .

For convenience, we extend the notation above as follows. We denote by h_0 the face of G to the left of e' when oriented from x' to y' , and we denote by h_m the face of G to the left of e when oriented from y to x . Next, consider the path from x to x' that bounds f and such that f lies on its right. Call the faces on the *left* side of this $x-x'$ path h_{m+1}, \dots, h_N . See Figure 3.

Letting $e_0 = e'$ and $e_m = e$, the angle between T and T' then is given by

$$\angle(T, T') = \sum_{i=0}^{m-1} \angle^{y_i}(e_{i+1}, e_i) \in \mathcal{S}(G),$$

where in each expression in the sum, we regard each edge as being oriented away from y_i in turn. Then by Lemma 3.2, we have

$$\phi(\angle(T, T')) = (f_1 - h_0) + (f_2 - h_1) + \dots + (f_{m-1} - h_{m-2}) + (f - h_{m-1}) \in \mathcal{S}(G^*).$$

The angle between T and T' is shown in Figure 5. The signs indicate $\phi(\angle(T, T')) \in \mathcal{S}(G^*)$.

Next, consider the oriented cycle C running from x' to y' , then along edges of T from y' to x , then along edges of f back to x' , as shown in Figure 6. The dual C^* of C is a cut of G^* , so $\partial_{G^*}(C^*) = 0 \in \mathcal{S}(G^*)$. On the other

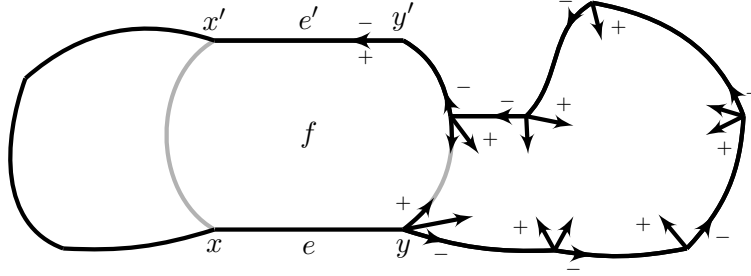


FIGURE 5. $\angle(T, T') \in \mathcal{S}(G)$ and $\phi(\angle(T, T')) \in \mathcal{S}(G^*)$, the former drawn with arrows and the latter drawn with plus and minus signs.

hand,

$$\partial_{G^*}(C^*) = (h_0 - f) + (h_1 - f_1) + \cdots + (h_{m-1} - f_{m-1}) + \sum_{i=m}^N (h_i - f).$$

The signs in Figure 6 indicate $\partial_{G^*}(C^*) \in \mathcal{S}(G^*)$.

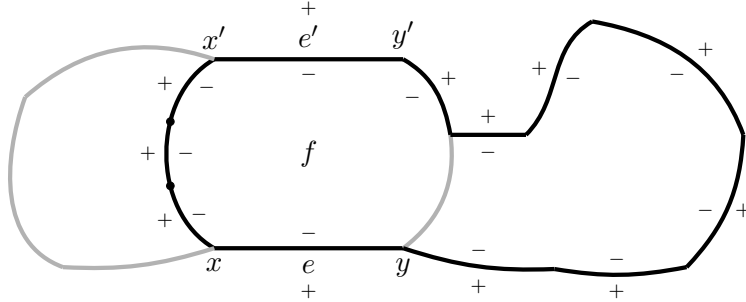


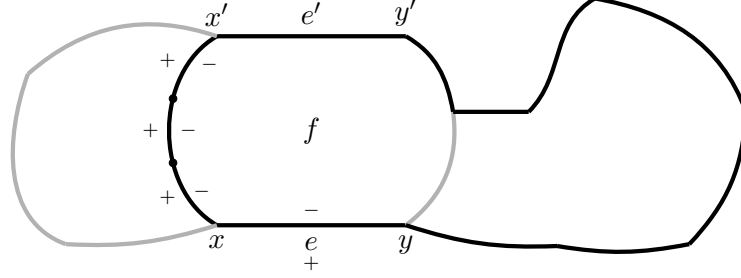
FIGURE 6. The cycle C in black and $\partial_{G^*}(C^*)$.

Summing, we have

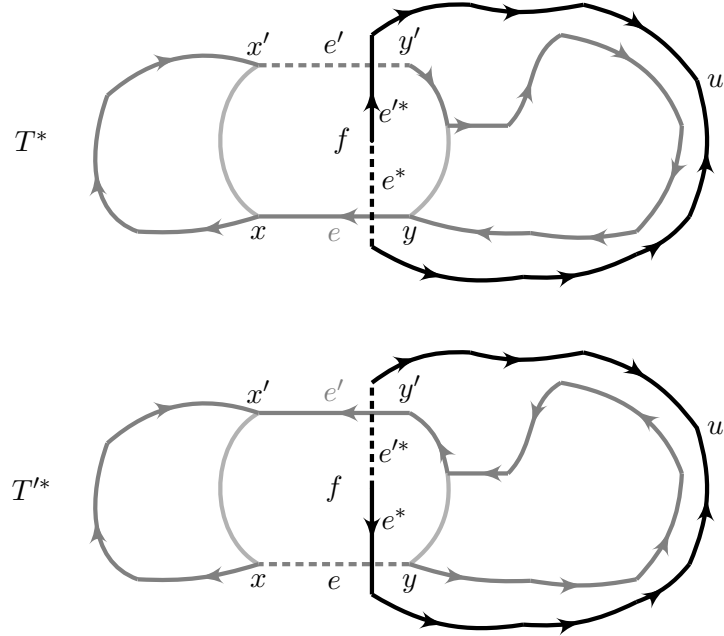
$$\phi(\angle(T, T')) + \partial_{G^*}(C^*) = \sum_{i=m}^N (h_i - f).$$

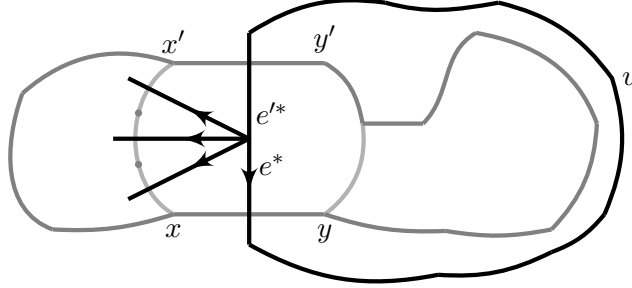
This sum is shown in Figure 7.

But this sum is exactly $\angle(T^*, T'^*)$. To see this, root the trees T^* and T'^* at a vertex u of G^* on the cycle in $T^* \cup \{e^*\}$ but different from f , as illustrated


 FIGURE 7. $\phi(\angle(T, T')) + \partial_{G^*}(C^*) \in \mathcal{S}(G^*)$.

in Figure 8. Then the only nonzero vertex angle contributing to $\angle(T^*, T'^*)$ is the angle at the vertex f , and by definition, this angle is $\sum_{i=m}^N (h_i - f)$, as shown in Figure 9. So we are done. \square


 FIGURE 8. Parts of the trees T^* and T'^* , rooted at u .

FIGURE 9. $\angle(T^*, T'^*)$.

We now prove our main result.

Proof of Theorem 3.1. Given $[D] \in \mathcal{S}(G)$ and $T \in \mathcal{T}(G)$, let $T' = [D] \cdot T$, and let $T'' = \phi([D]) \cdot T^*$. We would like to show that $T'' = T'^*$. By Lemma 3.4,

$$\phi(\angle(T, T')) = \phi([-D]) = \angle(T^*, T'').$$

By Lemma 3.7,

$$\phi(\angle(T, T')) = \angle(T^*, T'^*).$$

Hence, $\angle(T^*, T'') = \angle(T^*, T'^*)$. Therefore, $\angle(T'', T'^*) = 0$, and the result then follows from Corollary 3.5. \square

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